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A Note on Inherent Replication Properties of Local Cellular Automata Transition Functions

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Local transition functions of elementary cellular automata show different tendencies to replicate parts of a configuration in a later generation. This is seen as regularities in the time-space diagram. This replication depends on both the configuration and the local transition function. A possibility to isolate the influence of the local transition function is shown.

KEY WORDS: Cellular automata; discrete dynamical systems; elementary cellular automata; entropy.

1. INTRODUCTION

Let $\{0, 1\}^Z$ be the set of functions from the integers Z to the Boolean values $\{0, 1\}$; the elements of this set are also called "configurations." Elementary cellular automata are functions F from $\{0, 1\}^Z$ to $\{0, 1\}^Z$ with the further restriction that there is a *local transition function* f: $\{0, 1\}^3 \to \{0, 1\}$ such that the following formula hols:

$$\forall c \in \{0, 1\}^Z \, \forall z \in Z \quad F(c)(z) = f(c(z-1), c(z), c(z+1))$$

If c is a configuration and $c(z) = v$, we say that *cell z in configuration c has value v*.

The repeated application of F to an initial configuration c_0 yields a sequence $c_0, c_1, c_2,\ldots$ of configurations. The existence of the local transition function implies that the value of cell z in configuration c_i, $i > 0$, depends only on the value of cell z and its direct neighbors in configuration c_{i-1}, and the way it depends on these values is the same for all cells.

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It is easily seen that there are exactly 256 local transition functions for elementary cellular automata. They can numbered in the scheme of ref. 1; the function with number x will be denoted f_x in this paper. We will write the index in sedecimal notation.

If f: $\{0, 1\}^3 \to \{0, 1\}$ and $i \in N$, $i \geqslant 1$, then we denote by f^i the function which comprises the repeated application of f:

$$f^i\colon\quad \{0, 1\}^{2i+1} \to \{0, 1\}$$

$$f^i(k_{-i}, k_{-i+1}, \ldots, k_{i-1}, k_i)$$
$$= \begin{cases} f(k_{-1}, k_0, k_1) & \text{if} \quad i = 1 \\ f(f^{i-1}(k_{-i}, \ldots, k_{i-2}), \\ \quad f^{i-1}(k_{-i+1}, \ldots, k_{i-1}), \\ \quad f^{i-1}(k_{-i+2}, \ldots, k_i)) & \text{otherwise} \end{cases}$$

The effects of a repeated application of an elementary cellular automaton F to a given configuration in a finite range of cells can be visualized with a space/time diagram. The values of the finite range of cells are displayed as a sequence, and the corresponding finite sequences of states of succeeding configurations are dipslayed as a sequence of values in the next row. In Fig. 1 some examples are displayed, with dark spots for ones and light spots for zeros. In all these diagrams, the same initial configuration c_0 has been used. Furthermore, c_0 is spatial periodic with a period of 72, which means that for all $z \in Z$, $c_0(z) = c_0(z + 72)$. Because of this it suffices to display a range of 72 cells.

Diagrams like the one in Fig. 1 suggest the idea that with the application of some local transition functions, a configuration tends to be repeated or shifted after a fixed number of steps, whereas under the application of other transition functions, there is no repetition. This phenomenon depends presumably on both the local transition function and the initial configuration.

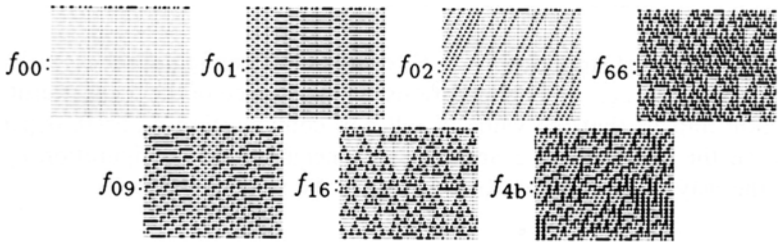

Fig. 1. Selected space/time diagrams with the same spatial periodic initial configuration and different local transition functions.

In this paper, the influence of the transition function alone is analyzed with the help of the entropy of regular languages.[2] It is demonstrated that the set of configurations which are reproduced by a given local transition function f in a given number g of steps with a given shift v to the left or to the right can be expressed as a regular language. The greater the entropy of this language, the greater the tendency of the local transition function to show repetition in the given number of steps and the given shift. This model of replication of configurations is called the one-phase model.

The appearance of some of the space/time diagrams is not very well explained by this model, because it does not take into account the possibility that under application of some local transition functions, the set of configurations which can occur after some steps is far smaller than the initial set of all configurations. If only a small set of configurations is repeated, but all other configurations, after a small number of steps, are transformed into one of these, the space/time diagram might look regular even if only a small number of configurations is repeated by the function. f_{00} is an extreme example of this case.

For these conditions, the entropies of another set of regular languages can be considered: These express the configurations which, under application of a given application function f, after a given number of steps g_0, yield configurations which are repeated in g_1 more steps with a shift of v. If many initial configurations yield repeating configurations, the corresponding entropies are big, even if of the initial configurations themselves only a few are repeated. This model of replications is called the two-phase model.

2. SIMILAR APPROACHES

Wolfram[3] considers the entropies (or, more exactly, the greatest eigenvalues) which can be associated to the regular languages describing the configuration a transition function of a cellular automaton can yield after a fixed and finite number of steps. This means that Wolfram is interested in the generability of configurations; this paper, in contrast, tries to characterize systematically fixed points of a given transition function.

Similar approaches look specifically and systematically at fixed points, but either restrict themselves to a special class of especially easily treatable transition function, the so-called "linear transition functions,"[10] or look only at space-periodic initial configurations (so that the configuration space is always finite and, after a finite transient, a limit cycle of configurations is reached),[4] make both restrictions,[5] or consider special transition functions which, in some sense, can be reduced to linear transition functions.[6]

This paper, in contrast, restricts its scope neither only to linear cellular automata nor only to space-periodic initial configurations. But another, regrettably quite severe restriction follows from this: In this work, we will not be able to characterize the limit sets of the considered cellular automata. For the general case, this is very plausible, since it has long been known that one-dimensional cellular automata with a state set and a neighborhood which are big enough (with some possible restrictions for one or the other) are, with a not too strange-looking definition of universal computability, computationally universal. But if the state set is restricted to two elements and the neighborhood to the described three, it is not known if there is an acceptable definition of universal computability that can be fulfilled by the model (and, what complicates the matter, "acceptable definition of universal computability" is not a very precise concept). Even disregarding the latter problem, we have to say that we do not know very much about the type of cellular automata considered in this paper. Perhaps more can be known about limit sets of this type of cellular automata.

3. THE ONE-PHASE MODEL

The one-phase model depends on the following theorem:

Theorem 3.1. Let f: $\{0, 1\}^3 \to \{0, 1\}$ be a local transition function of an elementary cellular automaton, F the corresponding global transition function, $g \in N$ a number of steps, $v \in Z$, $|v| \leqslant g$ a shift. Let $C(f, g, v) \subseteq \{0, 1\}^Z$ be the set of configurations which are repeated by application of f for g steps with a shift of v, or, more formally,

$$C(f, g, v) = \{c \in \{0, 1\}^Z\colon \forall z \in Z\ F^g(c)(z + v) = c(z)\}$$

Then the set L of all finite words formed by consecutive symbols of any configuration $c \in C(f, g, v)$ is a regular language.

Proof. For the proof, a finite automaton $A(f, g, v)$ which recognizes exactly L is constructed.[7] The construction is analogous to the construction of the finite automaton for the recognition of words in generable configurations[3]:

- The set of states T of the finite automaton is the set of $2g$-tuples of bits.
- The set E of input symbols for the finite automaton is the set of bits.
- For the definition of the transition function δ: $T \times E \to 2^T$ of the finite automaton we use a function (folg): $T \times E \to \{0, 1\}^{2g}$, with

$$\text{folg}(t, e) = (t_2, t_3, \ldots, t_{2g-1}, t_{2g-1}, t_{2g}, e)$$

with the abbreviation $t=(t_1, t_2, ..., t_{2g})$. For every state t and input symbol e, folg yields the next state. As another abbreviation, we use $w:=f^g(t_1, t_2, ..., t_{2g}, e)$ and $t_{2g+1}:=e$. We have

$$\delta(t, w)=\{\text{folg}(t, e)\colon e\in\{0, 1\}, w=t_{g+1-v}\}$$

We define recognition of a word by this nondeterministic finite automaton as the existence of a path whose labels build up the word.

It is easily seen that $A(f, g, v)$ recognizes exactly the finite subsequences of configurations which are repeated in the prescribed way. ■

By the construction in ref. 2, an entropy can be computed for the regular language recognized by $A(f, g, v)$. This entropy is the dyadic logarithm of the greatest eigenvalue of a matrix computed from the transition matrix of the deterministic version of the finite automaton $A(f, g, v)$. In this paper, we will give these greatest eigenvalues themselves, and not their logarithms. These eigenvalues can be interpreted as a mean of the number of possibilities in which a very long word which is in the recognized language can be extended by exactly one letter at one end so that the result again is a word of the language. These eigenvalues are therefore less than or equal to two, since this is the cardinality of the alphabet $\{0, 1\}$, and greater than or equal to zero. If the eigenvalue is zero, there is no word which is repeated with the given parameters; if the eigenvalue is one, there is just one possibility of extending a very long word; in ref. 7 is has been shown that in this case the repeated configurations are spatially periodic. And if

0x00:	1:				1	1	1			
	2:			1	1	1	1	1		
	3:		1	1	1	1	1	1	1	
	4:	1	1	1	1	1	1	1	1	1
01:	1:				0	0	0			
	2:			1	1	1.618	1	1		
	3:		1	1	0	0	0	1	1	
	4:	1	1	1	1	1.618	1	1	1	1
02:	1:				1.466	1	1			
	2:			1.466	1	1	1	1		
	3:		1.466	1	1	1	1	1	1	
	4:	1.466	1	1	1	1	1	1	1	1
66:	1:				1	1	1			
	2:			1	1	1	1	1		
	3:		1	1	1	1	1	1	1	
	4:	1	1	1	1	1	1	1	1	1
09:	1:				0	0	0			
	2:			1	1	1	1	1.38		
	3:		1	1.375	1	1	0	1	0	
	4:	1	1	1	1	1	1	1	1	1.38
16:	1:				1	1	1			
	2:			1	1	1	1	1		
	3:		1	1	1	1	1	1	1	
	4:	1	1.097	1	1	1.25	1	1	1.097	1
4b:	1:				1	1	0			
	2:			1	1	1	1	1		
	3:		1	1	1	1	1	1	1	
	4:	1	1	1	1	1	1	1	1	1

Fig. 2. Maximal eigenvalues of regular languages which represent sets of configurations which are repeated under a application of a given local transition function after a given number of steps with a given shift. The first numbers in the first lines are the sedecimal codes of the local transition functions, the values for the different shifts $-g, ..., g$ for a single number g of steps are given in a row. Compare Fig. 1.

the eigenvalue is two, then all configurations are repeated with the given parameters. All other eigenvalues lie between one and two.

If, for a number of steps, the maximal eigenvalue for one or two shifts is greater than for the other, we expect the corresponding movements to dominate in the space/time diagram of the cellular automaton. If all values are the same, we expect no movement to dominate; if eigenvalues are one or zero for a shift, we expect (almost) no noticeable movement with the corresponding parameters.

Figure 2 gives eigenvalues for the functions of Fig. 1 for $1 \leqslant g \leqslant 4$ and $-g \leqslant v \leqslant g$.

4. THE TWO-PHASE MODEL

The tables in Fig. 2 do not throw light on the dissimilarity of the space/time diagrams for functions f_{00} and f_{66}. Figure 1 makes it clear that it is desirable to discriminate between the functions. The problem with the eigenvalues of Fig. 2 is that it is not considered that under application of f_{00}, only one homogeneous configuration can occur after step 1, that this configuration is therefore repeated ever again, and that because of the homogeneity, it is repeated with all possible shifts. This suggests the development of a two-phase model: The series of configurations is split into two phases: a first phase of g_0 steps in which we are not interested in repetition phenomena but only in a possible restriction in the number of possible configurations, and a second phase in which we look again for repetition phenomena. For this method, we use the following theorem:

Theorem 4.1. Let f: $\{0, 1\}^3 \to \{0, 1\}$ be a local transition function of an elementary cellular automaton, F the corresponding transition function, $g_0, g_1 \in N$ numbers of steps, $v \in Z$, $|v| \leqslant g_1$ a shift. Let $C(f, g_0, g_1, v) \subseteq \{0, 1\}^Z$ be the set of configurations which, after g_0 applications of f, yield a configuration which is repeated by application of f for g_1 steps with a shift of v, or, more formally,

$$C(f, g_0, g_1, v) = \{c \in \{0, 1\}^Z\text{: } \forall z \in Z\; F^{g_1}(F^{g_0}(c))(z+v) = F^{g_0}(c)(z)\}$$

Then the set L of all finite words formed by consecutive symbols of any configuration $c \in C(f, g_0, g_1, v)$ is a regular language.

Proof. For the proof, a finite automaton $A(f, g_0, g_1, v)$ which recognizes exactly L is constructed. The construction is again analogous to the construction of the finite automaton for the recognition of words in generable configurations. Let $g = g_0 + g_1$.

00:	1:				2	2	2			
	2:			2	2	2	2	2		
	3:		2	2	2	2	2	2	2	
	4:	2	2	2	2	2	2	2	2	2
02:	1:				2	1.618	1.618			
	2:			2	1.618	1.618	1.618	1.618		
	3:		2	1.618	1.618	1.618	1.618	1.618	1.618	
	4:	2	1.618	1.618	1.618	1.618	1.618	1.618	1.618	1.618
09:	1:				0	0	0			
	2:			1.325	1.325	1.325	1.325	1.647		
	3:		1.144	1.609	1.091	1	0	1.272	0	
	4:	1.325	1.325	1.325	1.325	1.325	1.325	1.325	1.325	1.647

01:	1:				0	0	0			
	2:			1.839	1.839	2	1.839	1.839		
	3:		1	1	0	0	0	1	1	
	4:	1.839	1.839	1.839	1.839	2	1.839	1.839	1.839	1.839
66:	1:				1	1	1			
	2:			1	1	1	1	1		
	3:		1	1	1	1	1	1	1	
	4:	1	1	1	1	1	1	1	1	1
16:	1:				1.466	1.466	1.466			
	2:			1.466	1.466	1.466	1.466	1.466		
	3:		1.466	1.466	1.466	1.466	1.466	1.466	1.466	
	4:	1.466	1.466	1.466	1.466	1.466	1.466	1.466	1.466	1.466

4b:	1:				1	1	0			
	2:			1	1	1	1	1		
	3:		1	1	1	1	1	1	1	
	4:	1	1	1	1	1	1	1	1	1

Fig. 3. Maximal eigenvalues of regular languages which represent sets of configurations which, in two steps, are transformed into configurations which are repeated under application of a given local transition function after a given number of steps with a given shift. For the notation, compare Fig. 2.

- The set of states T of the finite automaton is the set of $2g$-tuples of bits.
- The set E of input symbols for the finite automaton is the set of bits.
- With the abbreviation $t = (t_1, t_2,..., t_{2g})$, we define the transition function δ of the finite automaton:

$$\delta(t, e') = \{(t_2,..., t_{2g}, e): e' = f^g(t_1,..., t_{2g}, e) \wedge e' = f^{g_0}(t_{1+g_1-v},..., t_{1+g_1-v+2g_0})\}$$

It is easily seen this nondeterministic finite automaton recognizes L. ■

For the automaton $A(f, g_0, g_1, v)$, we can again calculate the maximal eigenvalues of the transition matrix. In Fig. 3, the eigenvalues are given for the same parameters as in Fig. 2, with the exception that $g_0 = 2$. Considering these values, the difference between, for example, f_{00} and f_{66} becomes apparent.

The two-phase model, even if it seems to be superior to the one-phase model, has a defect: the choice of g_0, the number of steps which are made before the replication is considered, is more or less arbitrary. Our missing knowledge about limit sets of the considered type of cellular automata is the deeper source of this defect.

5. OTHER CHARACTERIZATIONS

Other properties of the finite automaton can be used for the characterization of the development of cellular automata in infinite configurations. The author[7] has given some characterizations of this type. They all consider the reduced and deterministic version of the constructed nondeterministic automata as described in this paper or by Wolfram[3]:

- The number of not-recognizing states is always zero or one; if it is zero, then the automaton has exactly one state (which is, of course, recognizing), and all configurations are recognized. If it is one, all transitions from the not-recognizing state go into this same state.
- For 72 of the considered transition functions, every initial configurations leads after at most two steps into a cycle of configurations which also has at most two steps. For all other $256 - 72 = 184$ transition functions, for every $g \in N$ there is an initial configuration such that even after g steps, no cycle of configurations is reached. The first part of this proposition can be deduced from the finite deterministic reduced versions of the automata constructed by Wolfram[3]: the automaton for two steps is the same as the one for four steps for 72 transition functions.
- If the number of transitions into recognizing states is equal to the number of recognizing states, then the set of limit configurations is finite. In this case, the restriction of the graph to recognizing states and transitions into recognizing states yields a number of disconnected subgraphs; this number is the number of different limit sets (not regarding differences through shifts).

The procedure which has been used in this paper has another character than the results listed above, because it abstracts from differences between formal languages which result from inclusion and exclusion of a finite number of finite words. It is hoped that this further abstraction, which I am perhaps allowed to call a more "statistical" approach, yields the structure of the set of transition functions in a more orderly way.

6. SUMMARY

In this paper the use of maximal eigenvalues of transition matrices for the description of inherent replication properties of local transition functions of elementary cellular automata was investigated. Two models were developed which allowed the characterization of different local transition functions.

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